J. Fluid Mech. (1968), *vol.* 32, *part* 2, *pp.* 317–331
Printed in Great Britain

The propagation upward of the shock wave from a strong explosion in the atmosphere

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(Received 9 May 1967 and in revised form 16 November 1967)

A method is established for the calculation of the trajectories of shocks moving upward in the atmosphere, on the basis of the assumption that they are of the self-propagating type. The results of calculations for self-similar motions are given, and these are used to establish a propagation law based upon the concepts of the Chisnell, Chester and Whitham (CCW) approximation. This propagation law enters a characteristics law based upon that proposed by Whitham, but reformulated for the computation of axisymmetric shocks with varying density.

An asymptotic self-preserving shock shape is investigated, and is computed for the case $\gamma = 1{\cdot}4$. A parabolic approximation scheme suggested by the self-preserving solution is developed, in which the solution near the axis is reduced to the solution of a system of ordinary differential equations. Finally, the governing equation for the general case without axial symmetry (but without winds) is presented.

1. Introduction

One of the problems encountered in the study of strong explosions in the lower atmosphere is that of computing an estimated time history of the shock wave, in particular as it travels upward into the ionosphere. Greatly simplified calculations (Kompaneets 1960; Andryankin *et al.* 1962) indicate that a strong explosion will 'vent' into the vacuum at infinite height within a finite time, and this conclusion is reinforced by an accurate theory (Raizer 1964) for one-dimensional strong shock propagation in an exponential atmosphere.

An explosion of moderate strength at moderately low altitude generally reaches a stage at which the shock is weak and the propagation is close to acoustic. The weakening of the shock to near-acoustic conditions is largely a consequence of the geometric divergence of the propagation rays, with the attendant increase in area of a ray tube. The parts of the shock wave that propagate upward pass into regions of ever-decreasing density, and this effect results in a strengthening of the shock. After the shock has traversed a couple of scale heights upward in the atmosphere, this effect overrides that of geometric divergence, and the shock can again become a strong one asymptotically. This strong shock must follow approximately the laws given by Raizer (1964) with errors arising from the non-uniformity of composition and scale height in the atmosphere and from the three-dimensionality of the shock shape.

We can distinguish two fairly distinct cases: (i) the case of a sufficiently weak explosion at sufficiently low altitude, and (ii) the case of a sufficiently strong explosion at sufficiently high altitude. In case (i) the propagation mechanism is basically that of Raizer; no material from the heart of the explosion is vented into space, and the main result of the venting of the shock surface into space is the establishment of a current of heated air upward. In case (ii) the propagation mechanism is basically that of Kompaneets and Andryankin *et al.*, with obvious adjustments for the limitations of the models; the bulk of the material from the heart of the explosion is vented into space in this case. In the present report we are concerned exclusively with case (i).

The aim of this report is to establish a computation scheme for upward-travelling shocks from explosions in case (i). A direct hydrodynamic calculation is not very attractive, because such a calculation would be three-dimensional (two space dimensions plus time). We wish a simpler scheme, but one of satisfactory engineering accuracy.

The shocks going upward are of the 'self-propagating' type after they have travelled far enough from the centre of the explosion to be accelerating. A shock is 'self-propagating' when its propagation is governed primarily by changes in the ray tube area and the gas state in front of the shock and is insensitive to conditions behind the shock. Such a self-propagating shock is characterized by the fact that a disturbance located farther than a certain critical distance (which should not be too great) behind the shock never catches up with the shock. Such a shock is also generally characterized by the fact that it is accelerating. In the more general non-self-propagating case, a shock may be influenced by any disturbance behind it.

An attractive approximation for the law governing self-propagating shocks is the CCW approximation, developed by Chisnell (1955, 1957), Chester (1954, 1960), and Whitham (1958). This approximation corresponds to applying to large changes in ray tube area or in the gas state in front of the shock the rule obtained by integrating the rule for infinitesimal changes. The error lies in the neglect of all information from behind the shock, whether this be reflexion of waves emitted from the shock or boundary conditions imposed in the downstream region. Whitham's rule for the approximation is that conditions on following characteristics (catching up with the shock) are to be applied immediately behind the shock itself, on the shock trajectory.

The agreement shown between the CCW approximation and precise calculations for implosions in a perfect gas has been so excellent as to be almost uncanny. For spherical imploding shocks and $\gamma = \frac{7}{5}$ or $\frac{5}{3}$, the agreement of the characteristic exponent is within 0·1 %. This agreement suggests that the approximation might work well also for density changes as well as for area changes. Accurate numerical results (Raizer 1964 and Hayes 1968) show that this is not generally the case. With an exponential density distribution, the CCW approximation gives values of the characteristic exponent that are in error by about 15 %. With a power-law density distribution (Sakurai 1960) the error is less.

What is suggested in place of the CCW law is a shock propagation law of the form suggested by the CCW approximation, but with exponents taken from accurate calculations of self-similar motions. The basic assumption in such a procedure is that of local similarity, an assumption long used in other approximate hydrodynamic calculations, particularly for boundary layers. Physically, this assumption should be better for self-propagating shocks than for boundary layers. The shock-propagation law may be simplified on a semi-empirical basis, provided it clearly reduces to correct form in certain limiting cases. These limiting cases must include the acoustic case and the strong shock case.

Once a satisfactory propagation law has been established, a spatial (axisymmetric or three-dimensional) calculation may be undertaken. The approach to be used is a modification of that of Whitham (1957, 1959), or an extension of that of Hayes (1963) for linear propagation. The essential feature here is that the geometry of the propagating shock governs the area changes in the ray tubes. These in turn govern or strongly influence the velocity history of the shock on each ray, and this history in turn governs the geometry of the propagating shock wave as a whole. Whitham has shown that the propagation law gives a definite velocity of lateral propagation of a disturbance along the shock surface, and this effect must be taken into account in any calculational scheme. In the present problem, only axisymmetric geometries are considered, and the final computation scheme is two-dimensional. The lateral disturbance propagation indicates real characteristics, and the mathematical system is hyperbolic. Either a proper characteristics method should be established, or a second-order derivative scheme that takes the existence of the characteristics into account and is stable.

Initial conditions must be taken so that the shock is initially of the self-propagating type. This requires that the results of an explosion calculation be available, in which the calculation has been carried out far enough that an appreciable part of the upper part of the shock has started to accelerate (because of the decreasing density upward).

We summarize the procedure as follows: (i) establish sufficient exact results for self-similar motions to serve as the basis for a shock-propagation law (Hayes 1968); (ii) establish a satisfactory shock-propagation law; (iii) establish a scheme for the calculation of axisymmetric shocks; (iv) apply the scheme to cases of interest, starting with an established shock of the self-propagating type.

We make a number of simplifying assumptions, primarily that there is no wind and that the gas is everywhere a calorically perfect gas. Neglected, of course, are any MHD or plasma effects.

2. Results for self-similar motions

The shock-propagation law desired is of the form

$$\frac{dU}{U} = \frac{M^2-1}{2M^2}\left\{-K(M,k,\gamma)\frac{d\rho}{\rho}\right\} \tag{2.1}$$

(in simplified form), where U is the velocity of propagation of the shock, ρ is the density in front of the shock, and $M = U/a_\infty$ is the Mach number of the shock (in front of the shock). The quantity

$$k = \frac{\rho}{A}\frac{dA}{d\rho} \tag{2.2}$$

is a dimensionless ratio of area change to density change. If K is linear in k, with

$$K = K_0 + K_1 k, \tag{2.3}$$

the law would take the form

$$\frac{dU}{U} = -\frac{M^2-1}{2M^2}\left\{K_0\frac{d\rho}{\rho} + K_1\frac{dA}{A}\right\}. \tag{2.4}$$

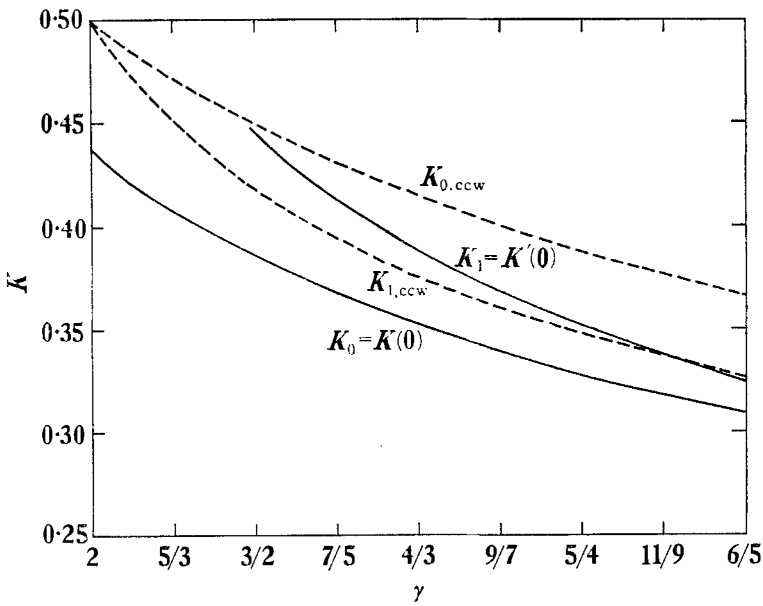

FIGURE 1. Dependence of K_0 and K_1 on γ.

This form is consistent with that given by the CCW approximation.

Self-similar solutions with strong shocks ($M \to \infty$) have been studied (Hayes 1968), with density and area following the laws

$$\rho = \rho_0 e^{-\beta z}, \tag{2.5a}$$

$$A = A_0 e^{-k\beta z}, \tag{2.5b}$$

where z is the altitude. In these cases the shocks do follow a law corresponding to (2.1),

$$\frac{dU}{U} = -\tfrac{1}{2}K(k)\frac{d\rho}{\rho}.$$

The results for $K_0 = K(0)$ are given in figure 1, together with results for $K_1 = K'(0)$ and, for comparison, the corresponding values given by the CCW approximation. The plot is against the parameter γ, but plotted so that the scale is proportional to $2/(\gamma-1)$. This choice yields smoother curves.

In figure 2 the variation of K with k is shown for the case $\gamma = 1{\cdot}4$, with parts of the corresponding curves for other values of γ, and the straight line $0{\cdot}394\,(1+k)$ which is expected to be an asymptote for large k. It will be observed that the curve for $\gamma = 1{\cdot}4$ is very close to a straight line. This observation supports the correctness of putting the strong-shock law into the form of (2.4).

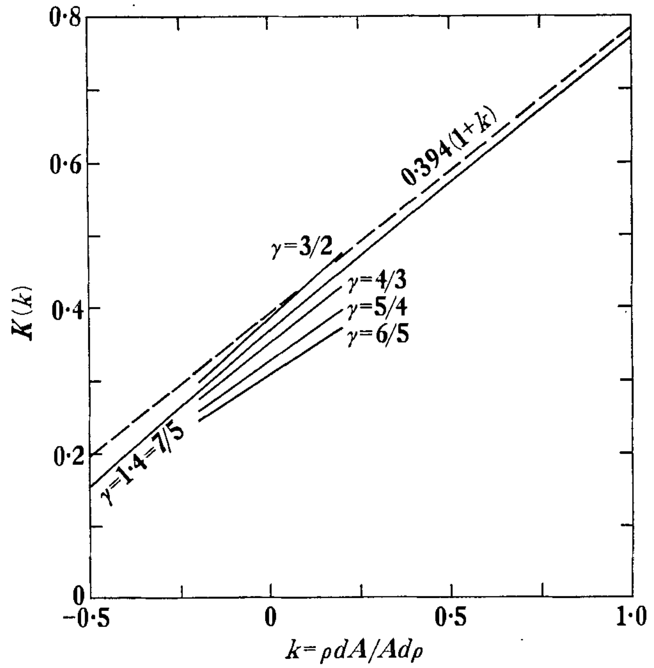

FIGURE 2. Dependence of K on k.

3. Propagation law

The propagation law for the shock wave is a law that gives changes in the velocity of the shock U under the assumption that these result only from changes in the density ρ in front of the shock, the area A of a ray tube, the undisturbed speed of sound a_∞, and the quantity $\frac{1}{2}(\gamma+1)$ for the gas. For simplicity the undisturbed gas is assumed to be at rest, with no winds. The parameters involved include parameters describing the relative magnitude of the various effects (such as the parameter k in (2.1)), plus the Mach number $M = U/a_\infty$.

The various effects are assumed to be independent, so that the respective terms may be separated (as in (2.4)). This separation leaves M (with γ, of course) as the basic parameter of the propagation law. This law must serve for almost acoustic waves as well as for strong shock waves, over the range of M from 1 to ∞.

In the acoustic range, $\rho u^2 A a_\infty$ is an invariant, and represents the flux of acoustic energy in a ray tube. Here u is the disturbance velocity, connected with U through the relation

$$U = \frac{\gamma+1}{4}u + \left[a_\infty^2 + \left(\frac{\gamma+1}{4}u\right)^2\right]^{\frac{1}{2}} \tag{3.1}$$

for a perfect gas. From the invariance of $\rho u^2 A a_\infty$, we have

$$\frac{du}{u} = -\frac{1}{2}\frac{d\rho}{\rho} - \frac{1}{2}\frac{dA}{A} - \frac{1}{2}\frac{da_\infty}{a_\infty}, \tag{3.2}$$

while from (3.1) we have

$$\frac{du}{u} = -\frac{d(\gamma+1)}{\gamma+1} + \frac{M^2+1}{M^2-1}\frac{dU}{U} - \frac{2}{M^2-1}\frac{da_\infty}{a_\infty}. \tag{3.3}$$

This gives the quasi-acoustic propagation law

$$\frac{dU}{U} = \frac{M^2-1}{M^2+1}\left\{-\frac{1}{2}\frac{d\rho}{\rho} - \frac{1}{2}\frac{dA}{A} + \frac{d(\gamma+1)}{\gamma+1} + \frac{1}{2}\left(\frac{4}{M^2-1} - 1\right)\frac{da_\infty}{a_\infty}\right\}. \tag{3.4}$$

This law may be altered by replacing 1 by M^2 in the coefficients without changing it essentially, except that the combination M^2-1 may not be changed.

For a strong shock, with area A held constant, the CCW procedure yields

$$\left[2+\left(\frac{2\gamma}{\gamma-1}\right)^{\frac{1}{2}}\right]\frac{dU}{U} = -\frac{d\rho}{\rho} + \left[1+\left(\frac{2\gamma}{\gamma-1}\right)^{\frac{1}{2}}\right]\left[\frac{d(\gamma+1)}{\gamma+1} + \frac{2}{M^2-1}\frac{da_\infty}{a_\infty}\right] \tag{3.5}$$

with the a_∞ term included approximately. As we have indicated, the CCW coefficient for density change we replace by that given by the self-similar analysis. We replace the ratio

$$\left[1+\left(\frac{2\gamma}{\gamma-1}\right)^{\frac{1}{2}}\right]\left[2+\left(\frac{2\gamma}{\gamma-1}\right)^{\frac{1}{2}}\right]^{-1}$$

by $\frac{3}{4}$, on the basis that the ratio is not too trustworthy except as giving an order of magnitude, and this order of magnitude is given as well by $\frac{3}{4}$. We thus get a strong-shock propagation law

$$\frac{dU}{U} = \frac{1}{2}\left\{-K_0\frac{d\rho}{\rho} - K_1\frac{dA}{A} + \frac{3}{2}\left[\frac{d(\gamma+1)}{\gamma+1} + \frac{2}{M^2-1}\frac{da_\infty}{a_\infty}\right]\right\}. \tag{3.6}$$

For a general Mach number, we need $K_0(M)$ and $K_1(M)$ in a law of the form of (2.4), with these parameters taking the values $\frac{1}{2}$ at $M = 1$. Here again we return to the CCW results for a guide, in lieu of much more extensive calculations of self-similar solutions. From figure 2 of Whitham (1957), an excellent fit is obtained through the relation

$$K_1(M) = K_1 + (\tfrac{1}{2} - K_1)M^{-2}. \tag{3.7}$$

This law for Mach number dependency is applied to our K_1, and also to our K_0. For the other coefficient, a fit between quasi-acoustic and strong-shock results is obtained by replacing the -1 following $4/(M^2-1)$ in (3.4) by $-M^{-2}$.

The resulting propagation law reads

$$\frac{dU}{U} = \frac{M^2-1}{2M^2}\left\{-[K_0+(\tfrac{1}{2}-K_0)M^{-2}]\frac{d\rho}{\rho} - [K_1+(\tfrac{1}{2}-K_1)M^{-2}]\frac{dA}{A} + \frac{3M^2+1}{2(M^2+1)}\left[\frac{d(\gamma+1)}{\gamma+1} + \frac{2}{M^2-1}\frac{da_\infty}{a_\infty}\right]\right\}. \tag{3.8}$$

This law has the correct form for both the quasi-acoustic and strong-shock cases, and the behaviour of the coefficients for intermediate Mach number is close to what may be expected on the basis of available information.

How should the law be applied best if the gas is not a perfect gas? The parameter γ enters (3.8) in two ways, as a parameter determining K_0 and K_1, and as a variable in the expression $d(\gamma+1)/(\gamma+1)$. In both ways the effect is small in the quasi-acoustic range and appreciable only with M^2-1 appreciable. In the effect of γ as a parameter, the key quantity is the limiting Mach number behind a strong shock,

$$M_s = \left(\frac{\gamma-1}{2\gamma}\right)^{\frac{1}{2}} \tag{3.9}$$

for a perfect gas. If desired, a γ for the purpose of estimating K_0 and K_1 could be obtained through (3.9) after calculating M_s from

$$M_s^2 = \frac{\epsilon_{1\text{lim}}\, p_s}{(1-\epsilon_{1\text{lim}})\rho_s a_s^2} \tag{3.10}$$

with sub s denoting conditions behind the actual shock, and

$$\epsilon_{1\text{lim}} = \frac{p_s/\rho_s}{h_s+e_s}. \tag{3.11}$$

In the effect of $(\gamma+1)$ as a variable, its function is through the relation

$$1-\epsilon_{1\text{lim}} = \frac{2}{\gamma+1} \tag{3.12}$$

for a perfect gas. Thus we should replace $\gamma+1$ by

$$\gamma+1 = \frac{h_s}{e_s}+1 \tag{3.13}$$

for the purpose of simulating the effect of this variable.

In the sequel, the quantities ρ, a_∞ and $\gamma+1$ are considered to be functions of the altitude z alone. The quantity

$$-\frac{1}{\rho}\frac{d\rho}{dz} = \beta \tag{3.14}$$

is the inverse of the classical scale height in the atmosphere, itself a function of z. For the purpose of simplifying the form of the propagation law, we introduce the following notation:

$$\mu^2 = \frac{M^2-1}{2M^2}\left[K_1+(\tfrac{1}{2}-K_1)M^{-2}\right], \tag{3.15}$$

$$F(M,z) = \frac{M^2-1}{2M^2}\left\{[K_0+(\tfrac{1}{2}-K_0)M^{-2}]\beta + \frac{3M^2+1}{2(M^2+1)}\left[\frac{1}{\gamma+1}\frac{d(\gamma+1)}{dz}+\frac{2}{M^2-1}\frac{da_\infty}{a_\infty dz}\right]\right\}. \tag{3.16}$$

The propagation law then takes the form

$$\frac{dU}{U} = -\mu^2\frac{dA}{A}+F\,dz \tag{3.17}$$

in our layered atmosphere. The quantity $\mu(M,z)$ will represent a characteristic slope in the sequel.

4. Axisymmetric calculation

Our problem is the establishment of a calculation procedure for the propagation of an axisymmetric shock of self-propagating type upward in the atmosphere. This procedure may be of one of three types: (i) a direct calculation based upon the approach of Hayes (1963) in which the ray tube divergence D is calculated along ray tubes. This approach does not directly utilize the characteristics noted by Whitham (1957), and is not chosen. (ii) A direct characteristics calculation based upon the work of Whitham (1957), with appropriate modifications to take the axial symmetry and varying density into account. This approach seems to be the simplest and most direct. (iii) A calculation using the time t as

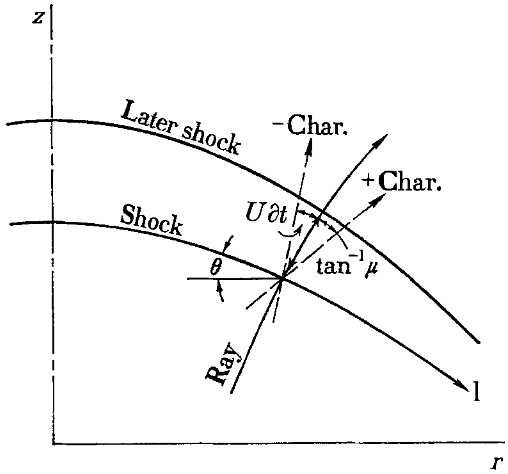

FIGURE 3. Shock geometry.

the basic dependent variable, using a second-order hyperbolic partial differential equation in the two space co-ordinates, following Whitham (1959) with appropriate modifications. This approach is not chosen, but is outlined in §7.

A ray is a normal trajectory (with no winds) to the family of shock surfaces. The inclination of a ray with the vertical, or of the shock surface with the horizontal, is denoted θ (see figure 3).

The curvature $\mathbf{K}$ of a ray obeys the law (Hayes 1963, equation (4.9); or Whitham 1957, equation (6))

$$\mathbf{K} = -\frac{\nabla_t U}{U} = -\frac{1}{U}\frac{\partial U}{\partial l}\,\mathbf{e}_l. \tag{4.1}$$

In the axisymmetric case, we have

$$\frac{\partial \theta}{U\partial t} = -\frac{1}{U}\frac{\partial U}{\partial l}, \tag{4.2}$$

where $\partial/\partial t$ is a time derivative along a ray, and $\partial/\partial l$ is an instantaneous tangential derivative along the shock profile; $\mathbf{e}_l$ in (4.1) is a unit vector tangent to the shock.

The divergence tensor $\mathbf{D}$ is a 2-2 tensor in the shock surface, essentially the curvature tensor for the surface. This quantity obeys the law

$$\frac{\partial \mathbf{D}}{U\partial t} + \mathbf{D}\cdot\mathbf{D} = -\frac{1}{U}\nabla_t\nabla_t U \tag{4.3}$$

(Hayes 1963, equations (5.3) and (5.4)), while the area change along a ray tube is given by

$$\frac{1}{A}\frac{\partial A}{U\partial t} = \operatorname{tr}\mathbf{D} \tag{4.4}$$

(Hayes 1963, equation (5.2)). In the axisymmetric case the natural axes are the principal axes for **D**, which then has principal components

$$\operatorname{diag} \mathbf{D} = \left\{D, \frac{\sin\theta}{r}\right\} \tag{4.5}$$

in cylindrical co-ordinates (r, z), with

$$D = \frac{\partial\theta}{\partial l} = \frac{d\sin\theta}{dr}. \tag{4.6}$$

The formula for area change, equation (4.4), then reads

$$\frac{1}{A}\frac{\partial A}{U\partial t} = \frac{\partial\theta}{\partial l} + \frac{\sin\theta}{r} = \frac{d\sin\theta}{dr} + \frac{\sin\theta}{r} \tag{4.7}$$

(compare Whitham 1957, equation (5)).

In our chosen approach (ii) we do not use (4.3), but instead combine (4.2) and (4.7) to obtain characteristic equations. We multiply (4.7) by $\mu^2 U\,dt$ and eliminate $d \ln A/U dt$ by means of the propagation law (3.17). We multiply (4.2) by dl, and add. The result is

$$\frac{\partial \ln U}{U\partial t}U\,dt + \frac{\partial \ln U}{\partial l}dl + \frac{\partial\theta}{U\partial t}dl + \mu^2\frac{\partial\theta}{\partial l}U\,dt + \left(\mu^2\frac{\sin\theta}{r} - F\cos\theta\right)U\,dt = 0, \tag{4.8}$$

in which the relation $\partial z/U\partial t = \cos\theta$ has been used. The characteristics are the directions along which the partial derivatives in (4.8) may be replaced by ordinary differentials. They are given by

$$\frac{dl}{U\,dt} = \pm\mu, \tag{4.9}$$

for which (4.8) gives the characteristic relations

$$\frac{d\ln U}{\mu} \pm d\theta + \frac{1}{\mu}\left(\mu^2\frac{\sin\theta}{r} - F\cos\theta\right)U\,dt = 0. \tag{4.10}$$

In this analysis l has essentially the function of an intrinsic co-ordinate, not of a co-ordinate in physical space. The differential dl is equivalent to Whitham's $A\,d\beta$, where β is a co-ordinate specifying the ray. Whitham's A is a ray tube area in the planar case he treats, and acts as a metric coefficient; its use is avoided in our analysis.

In applying the characteristic relations to a problem, adequate initial conditions specifying U and θ for a given initial shock shape must be given. As mentioned before, the shock initially should be known to be of the self-propagating type. Boundary conditions are two in number. One boundary condition is really a regularity condition, and is simply that

$$\theta = 0 \quad \text{on} \quad r = 0. \tag{4.11}$$

The other condition is imposed at the shoulder of the shock, and should be chosen in such a way that it interferes the least with the development of the shock over its main central portion. It is possible that 'shock-shocks' (Whitham 1957) may appear near the shock shoulder.

The calculation is analogous to that for an axisymmetric steady potential gas flow with sources (corresponding to the F term). Individual rays need not be kept identified, as no property is convected directly along rays. Rays may be obtained subsequently by integrating $dr/dz = \tan\theta$. Analogously, it is not necessary to keep track of shock shape or of the time t. Shock shapes may be obtained by integrating $dz/dr = -\tan\theta$, or by calculating time and exhibiting constant-time surfaces.

Characteristic calculations using (4.10) have not as yet been carried out. We turn next to the study of a particular self-similar solution and of an approximate scheme suggested by the self-similar solution.

5. Self-similar shock shapes

A plane shock moving upward in the atmosphere may readily be shown to be unstable, so that planar shocks cannot exist. The question naturally arises as to whether curved shock shapes of stationary form exist, which can then be expected to represent the asymptotic shape of actual shocks in an exponential atmosphere. These have been investigated under the assumption that the shape is axisymmetric.

The co-ordinates z and r are defined as before, and the co-ordinate x is directed downward from the vertex (or apex) of the shock. The upward velocity of the entire shock shape is $U_0(t)$. The moving co-ordinate x is thus given by

$$x = \int U_0 dt - z. \tag{5.1}$$

With the strong-shock approximation, $\mu^2 = \frac{1}{2}K_1$ and $F = \frac{1}{2}K_0\beta$. The parameters $(\gamma+1)$ and β are considered constant. From (4.7) and (3.17) we have

$$\frac{1}{2}K_1\left(\frac{d\sin\theta}{dr}+\frac{\sin\theta}{r}\right) = -\frac{d\ln U}{U dt}+\frac{1}{2}K_0\beta\frac{dz}{U dt}. \tag{5.2}$$

The quantity $dz/U\,dt$ is replaced by $\cos\theta$. The normal velocity of the shock U is given by

$$U = U_0\cos\theta \tag{5.3}$$

while the quantity

$$\frac{d\ln U_0}{\beta U_0 dt} = \alpha^{-1} \tag{5.4}$$

is assumed constant. This assumption is suggested by the solutions of the preceding paper of Hayes (1968).

The result of substituting (5.3) and (5.4) into (5.2) with the relation

$$dr = \sin\theta U dt, \tag{5.5}$$

is the ordinary differential equation

$$\frac{\frac{1}{2}K_1-\tan^2\theta}{1+\tan^2\theta}\frac{d\tan\theta}{d\beta r}+\frac{1}{2}K_1\frac{\tan\theta}{\beta r}-\frac{1}{2}K_0+\alpha^{-1}(1+\tan^2\theta) = 0. \tag{5.6}$$

The boundary condition at $\beta r = 0$ is that $\tan\theta = 0$ there, and that the solution be regular. There is a critical point at $\tan\theta = (\frac{1}{2}K_1)^{\frac{1}{2}}$, and the second boundary condition is that the solution pass through a saddle point there. This condition determines the correct value of α. Solutions that do not pass through a saddle point do not have both βr and $\tan\theta$ increasing monotonically.

This equation has been solved numerically for the case

$$\gamma = 1{\cdot}4, \text{ with } \tfrac{1}{2}K_1 = 0{\cdot}207 \text{ and } \tfrac{1}{2}K_0 = 0{\cdot}183.$$

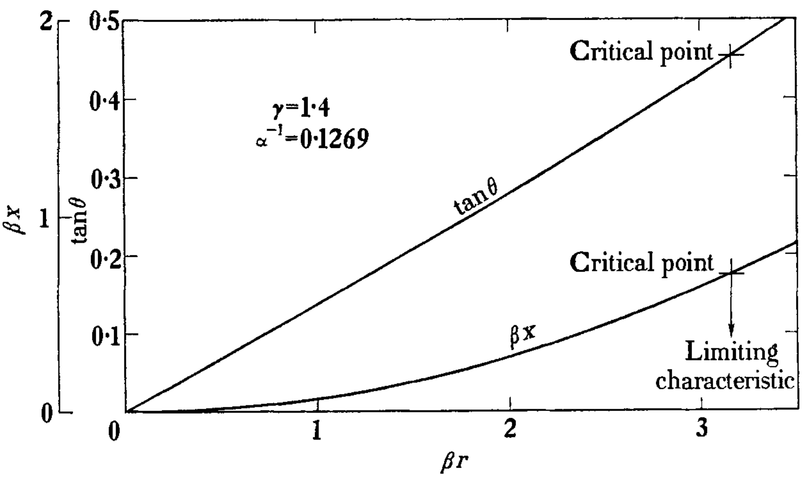

FIGURE 4. Self-similar shock shape ($\gamma = 1{\cdot}4$).

It was found that $\alpha^{-1} = 0{\cdot}1269$ for this case. The self-similar shape obtained is given in figure 4, with βx obtained from the relation

$$\frac{d\beta x}{d\beta r} = \tan\theta. \tag{5.7}$$

It turns out to be close to a paraboloid of revolution. This observation is exploited in the following section.

This calculation confirms the existence of such shapes, and gives us some idea of the extent of initial data needed in a complete characteristics calculation. Initial data should be available to $\tan\theta \approx 0{\cdot}6$ (or to $\beta r \approx 4$), at least.

The critical point is to be interpreted as corresponding to a limiting characteristic that is vertical. All characteristics from any part of the shock outboard of the critical point go farther away from the centre. From any point inboard of the critical point, one characteristic moves toward the centre and eventually intersects the axis $r = 0$.

6. Parabolic approximation

The closeness of the computed profile of figure 4 to a parabola suggests the possibility of a further simplification. The substitution

$$\tan\theta = C\beta r \tag{6.1}$$

in (5.6) is made. At $\theta = 0$ we obtain the condition

$$\alpha^{-1} = \tfrac{1}{2}K_0 - K_1 C. \tag{6.2}$$

With C equal to $d\tan\theta/d\beta r$ at $\theta = 0$ this condition holds in general. To evaluate C we require (5.6) with (6.1) to hold at the critical point, at $\tan^2\theta = \frac{1}{2}K_1$. With (6.2) we obtain

$$C = \alpha^{-1}, \tag{6.3}$$

and then

$$\alpha^{-1} = \frac{\frac{1}{2}K_0}{1+K_1}. \tag{6.4}$$

In the numerical example considered above this formula yields $\alpha^{-1} = 0{\cdot}1294$, with an error of 2 % from the computed value of 0·1269.

We are led to inquire as to why this very simple calculation gives such a close result for the propagation constant. One reason is that the parameter $\frac{1}{2}K_1 = 0{\cdot}207$ is appreciably smaller than unity. The suggestion is that (6.1) represents a power series truncated at one term, and that the smallness of $\frac{1}{2}K_1$ should be used in justifying the truncation.

Simply expanding (5.6) in powers of $\tan\theta$ and substituting (6.1) leads to a formula different from (6.4), yielding $\alpha^{-1} = 0{\cdot}1362$, with an error of 7 %. Thus the key to the close result must lie in the particular choice of the method of accomplishing the truncation, with the second condition one applied at or near the critical point. We next develop a method for the general case based upon this idea.

The shock shape is assumed to be given by

$$z = z_0(t) - \tfrac{1}{2}z_1(t)r^2, \tag{6.5}$$

and the slope accordingly by

$$\tan\theta = -\frac{dz}{dr} = z_1(t)r. \tag{6.6}$$

The quantity z_1 has the dimensions of inverse distance, and is identified as $C\beta$ in the approximation (6.1) for the self-preserving solution. From (4.7) we obtain the area derivative

$$\frac{1}{A}\frac{\partial A}{U\partial t} = (1+\cos^2\theta)z_1\cos\theta, \tag{6.7}$$

with ∂ indicating differentials along a ray. We have also

$$\frac{\partial r}{U\partial t} = \sin\theta; \quad \frac{\partial z}{U\partial t} = \cos\theta. \tag{6.8}$$

The propagation velocity of the shock is obtained by differentiating (6.5) along a ray. The result is

$$U = (z_0' - \tfrac{1}{2}z_1' r^2)\cos\theta, \tag{6.9}$$

for which (5.3) represents a special case. We differentiate (6.9) along a ray, to obtain

$$\frac{\partial U}{U\partial t} = \frac{1}{U}(z_0'' - \tfrac{1}{2}z_1'' r^2 - z_1' rU\sin\theta)\cos\theta - \sin\theta\cos\theta\,(z_1' r + z_1 U\sin\theta). \tag{6.10}$$

In obtaining (6.10), the derivative of (6.6) along a ray was used.

The shock propagation law (3.17) is now invoked, with the result

$$z_0'' - \tfrac{1}{2}z_1'' r^2 - 2z_1' rU\sin\theta - z_1 U^2\sin^2\theta = -\mu^2(1+\cos^2\theta)z_1 U^2 + FU^2. \tag{6.11}$$

At $\theta = 0$ this yields the equation

$$z_0'' = (F_0 - 2\mu_0^2 z_1) z_0'^2, \tag{6.12}$$

which can be identified with (6.2) by identifying $z_0''/z_0'^2$ as $\beta\alpha^{-1}$ through (5.4). In (6.12), μ_0 and F_0 designate the quantities μ and F evaluated at $(z, r) = (z_0, 0)$. If z_1 can be considered as a known function of z_0 and z_0', the solution of (6.12) gives the shock trajectory directly.

To carry out the truncation, we intend to satisfy (6.11) approximately at $\tan\theta = \mu_0$. Satisfying (6.11) exactly at $\tan\theta = \mu_0$ leads to

$$\frac{1}{2}\frac{z_1''}{z_1^2} + 2\frac{z_1'}{z_1}\left(z_0' - \frac{z_1'\mu_0^2}{4z_1^2}\right)\left(\frac{1}{1+\mu_0^2} + \mu_0^2 - \frac{F_0}{2z_1}\right) - \frac{1}{2}\frac{z_1'^2}{z_1^3}\frac{\mu_0^2}{1+\mu_0^2}$$
$$+ U^2\left\{z_1\left(1 + 2\mu_0^2 - \frac{F_0}{z_1}\right) - z_1\left(\frac{\mu^2 - \mu_0^2}{\tan^2\theta}\right)\left(\frac{2+\mu_0^2}{1+\mu_0^2}\right) + \left(\frac{F - F_0}{\tan^2\theta}\right)\right\} = 0, \tag{6.13}$$

obtained by subtracting (6.11) from (6.12). To simplify this expression, we neglect $z_1'\mu_0^2/z_1^2 z_0'$ in comparison with unity, and drop the factor $U^2/z_0'^2$. The terms in $\mu^2 - \mu_0^2$ and $F - F_0$ we replace by terms obtained from Taylor series. We obtain thereby

$$\frac{1}{2}\frac{z_1''}{z_1^2} + 2\frac{z_1' z_0'}{z_1}\left(1 - \frac{F_0}{2z_1}\right) + z_1 z_0'^2\left(1 + 2\mu_0^2 - \frac{F_0}{z_1}\right)$$
$$- 2z_1 z_0'^2\left(\frac{d\mu^2}{d\tan^2\theta}\right)_0 + z_0'^2\left(\frac{dF}{d\tan^2\theta}\right)_0 = 0. \tag{6.14}$$

From either (6.13) or (6.14), if $z_1' \equiv 0$ and μ^2 and F are constant, we obtain

$$z_1 = \frac{F_0}{1 + 2\mu_0^2}. \tag{6.15}$$

Together with (6.12) this gives $z_0''/z_0'^2 = z_1$, the same result as that of (6.3) and (6.4). Thus the approximate calculation for the self-preserving solution is in accord with our more general parabolic truncation.

The quantities μ^2 and F are both functions of M and z. We may write

$$\left(\frac{1}{U}\frac{dU}{d\tan^2\theta}\right)_0 = -\frac{1}{2}\left(1 + \frac{z_1'}{z_0' z_1^2}\right) \tag{6.16}$$

from (6.9), and

$$\frac{dz}{d\tan^2\theta} = \frac{1}{2z_1} \tag{6.17}$$

from (6.5) and (6.6). We then have

$$\left(\frac{dF}{d\tan^2\theta}\right)_0 = \left(M\frac{\partial F}{\partial M}\right)_0\left(\frac{1}{U}\frac{dU}{d\tan^2\theta}\right)_0 + \left(\frac{\partial F}{\partial z} - M\frac{\partial F}{\partial M}\frac{da_\infty}{a_\infty dz}\right)_0\frac{dz}{d\tan^2\theta}$$
$$= -\frac{1}{2}\left(1 + \frac{z_1'}{z_0' z_1^2}\right)\left(M\frac{\partial F}{\partial M}\right)_0 + \frac{1}{2z_1}\left(\frac{\partial F}{\partial z} - M\frac{\partial F}{\partial M}\frac{da_\infty}{a_\infty dz}\right)_0. \tag{6.18}$$

Similarly, we express

$$\left(\frac{d\mu^2}{d\tan^2\theta}\right)_0 = -\frac{1}{2}\left(1 + \frac{z_1'}{z_0' z_1^2}\right)\left(M\frac{\partial\mu^2}{\partial M}\right)_0 + \frac{1}{2z_1}\left(\frac{\partial\mu^2}{\partial z} - M\frac{\partial\mu^2}{\partial M}\frac{da_\infty}{a_\infty dz}\right). \tag{6.19}$$

These expressions are to be substituted into (6.14). The two equations (6.12) and (6.14) form a fourth-order ordinary differential system for calculating the functions $z_0(t)$ and $z_1(t)$. Initial data must include initial values of z_0, z_0', z_1 and z_1'.

A very rough perturbation analysis of (6.14) in the strong shock range, with μ_0^2 dropped in comparison with unity and $z_0(t)$ unperturbed, leads to a solution of the form

$$\delta z_1 = A\, e^{-z_0 F_0} + B\, e^{-2z_0 F_0}. \tag{6.20}$$

This shows that the relaxation distance for a shock trajectory with respect to the approach of z_1 to its asymptotic value is of the order of F_0^{-1}, or of the order of 6 scale heights.

7. General case

In the general case, without axial symmetry, we must follow a procedure of type (iii) (as discussed at the beginning of §4). For completeness, we outline this procedure here.

The time $t(x, y, z)$ is chosen as the primary dependent variable. The surfaces of constant t are the wave fronts, for which the normal vector is

$$\mathbf{n} = \frac{\nabla t}{|\nabla t|}. \tag{7.1}$$

The area integral is given (*cf.* (4.4)) by

$$\frac{1}{A}\frac{\partial A}{\partial n} = \operatorname{tr}\mathbf{D} = \operatorname{tr}\nabla\mathbf{n} = \nabla\cdot\mathbf{n} = \frac{\nabla^2 t}{|\nabla t|} - \frac{1}{|\nabla t|^3}\nabla t\cdot\nabla\frac{(\nabla t)^2}{2}. \tag{7.2}$$

The velocity U is simply equal to $|\nabla t|^{-1}$, and the derivative along the ray is

$$\frac{1}{U}\frac{\partial U}{\partial n} = \frac{1}{U}\mathbf{n}\cdot\nabla U = -\frac{1}{|\nabla t|^3}\nabla t\cdot\nabla\frac{(\nabla t)^2}{2}. \tag{7.3}$$

The derivative $\partial z/\partial n$ is given by

$$\frac{\partial z}{\partial n} = \cos\theta = \mathbf{n}\cdot\mathbf{k} = \frac{\partial t/\partial z}{|\nabla t|}. \tag{7.4}$$

We now invoke the propagation law (3.17), and multiply through by $|\nabla t|$. The result is the hyperbolic differential equation

$$\mu^2\nabla^2 t - \frac{1+\mu^2}{(\nabla t)^2}\nabla t\cdot\nabla\frac{(\nabla t)^2}{2} = F\frac{\partial t}{\partial z}. \tag{7.5}$$

The Mach number M, on which μ^2 and F depend, is given simply by

$$M = \left[\frac{1}{(\nabla t)^2 a_\infty^2}\right]^{\frac{1}{2}} = \frac{1}{|\nabla t| a_\infty}. \tag{7.6}$$

Equation (7.5) is closely analogous to the potential equation for three-dimensional steady supersonic flow. An essential difference is that (7.5) includes a source term, the one in F. For more details on the analogy (without the source term) see Whitham (1959). The equivalent 'Mach' cone angle, as expected, is simply $\tan^{-1}\mu$. The quantity here which is analogous to the irrotational velocity vector in steady supersonic flow is the *inverse* velocity vector $\nabla t = U^{-1}\mathbf{n}$. The quantity here which is analogous to the square of the speed of sound is

$$\mu^2(\nabla t)^2/(1+\mu^2) = \mu^2/(1+\mu^2)U^2.$$

The quantity which is analogous to the square of the steady flow Mach number is $1+\mu^{-2}$.

This work was performed with the support of the U.S. Office of Naval Research, under Contract Nonr. 4913 (00), ARPA Order 215.

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